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On Hamilton-Jacobi-Bellman-Isaacs Equation for Time-Delay Systems ⋆

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Abstract: The paper deals with a two-person zero-sum differential game for a dynamical system which motion is described by a delay differential equation under an initial condition defined by a piecewise continuous function. For the value functional of this game, we derive the Hamilton-Jacobi type equation with coinvariant derivatives. It is proved that, if the solution of this equation satisfies certain smoothness conditions, then it coincides with the value functional. On the other hand, it is proved that, at the points of coinvariant differentiability, the value functional satisfies the derived Hamilton-Jacobi equation. Therefore, this equation can be called the Hamilton-Jacobi-Bellman-Isaacs equation for time-delay systems.

Keywords: delay system, differential game, Hamilton-Jacobi equation, coinvariant derivatives, value functional, optimal strategies.

1. INTRODUCTION

From the differential games theory for ordinary differential equations (see, e.g., Isaacs (1965); Krasovskii and Subbotin (1988); Osipov (1971)) and Hamilton-Jacobi (HJ) equations with partial derivatives theory (see, e.g., Subbotin (1995); Crandall and Lions (1983); Clarke, Ledyaev, Stern and Wolenski (1998)), it is well known that, on the one hand, a value function of a differential game at points of differentiability satisfies the corresponding Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation with partial derivatives and, on the other hand, the continuously differentiable solution of the HJBI equation coincides with the value function. In Lukoyanov (2003) the similar result for differential games for time-delay systems and the corresponding HJBI equations with coinvariant derivatives was obtained. Note that HJBI equations were considered on the space of continuous functions. In the paper the similar result in the more general case for HJBI equations on the space of piecewise continuous functions is proved. Firstly, it allows to construct positional optimal strategies of players in differential games for time-delay systems, in the cases when the corresponding HJBI equation has a smooth solution even if an initial motion history has points of discontinuities. Secondly, the choice of the space of piecewise continuous functions may in the future significantly facilitate many constructions of the theory of generalized solutions of HJ equations with coinvariant derivatives. Earlier, similar investigations of HJ equations corresponding to optimal control problems for time-delay systems were presented in Plaksin (2019). The paper continues Plaksin (2019) and gives a theoretical foundation for further investigations of differential games for time-delay systems and the corresponding HJBI equations.

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2. DIFFERENTIAL GAME

Let $\mathbb{R}^n$ be the n-dimensional Euclidian space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. A function $x(\cdot)\colon [a,b] \mapsto \mathbb{R}^n$ is called piecewise continuous if there exist numbers $a = \xi_1 < \xi_2 < \ldots < \xi_k = b$ such that, for each $i \in \overline{1, k-1}$, the function $x(\cdot)$ is continuous on the interval $[\xi_i, \xi_{i+1})$ and there exists a finite limit of $x(\xi)$ as ξ approaches ξ_{i+1} from left. Denote by $\mathrm{PC}([a,b],\mathbb{R}^n)$ and $\mathrm{Lip}([a,b],\mathbb{R}^n)$ the linear spaces of piecewise continuous and Lipschitz continuous functions $x(\cdot)\colon [a,b] \mapsto \mathbb{R}^n$.

Let $t_0 < \vartheta$ and $h > 0$. Let us denote

$$\mathrm{PC} = \mathrm{PC}([-h,0],\mathbb{R}^n), \quad \mathbb{G} = [t_0,\vartheta] \times \mathbb{R}^n \times \mathrm{PC}.$$

Define the following norms on the space PC:

$$\|w(\cdot)\|_1 = \int_{-h}^{0} \|w(\xi)\| \mathrm{d}\xi, \quad \|w(\cdot)\|_\infty = \sup_{\xi \in [-h,0]} \|w(\xi)\|.$$

We consider a two-person differential game for the dynamical system described by the time-delay equation

$$\begin{gathered} \dot{x}(\tau) = f(\tau, x(\tau), x(\tau-h), u(\tau), v(\tau)), \\ \tau \in [t_0,\vartheta], \quad x(\tau) \in \mathbb{R}^n, \quad u(\tau) \in \mathbb{U}, \quad v(\tau) \in \mathbb{V}, \end{gathered} \tag{1}$$

and the quality index

$$\begin{aligned} \gamma = {} & \sigma(x(\vartheta), x_\vartheta(\cdot)) \\ & + \int_t^\vartheta f^0(\xi, x(\xi), x(\xi-h), u(\xi), v(\xi))\, \mathrm{d}\xi. \end{aligned} \tag{2}$$

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Here τ is the time variable; $x(\tau)$ is the value of the state vector at the time τ; $t \in [t_0, \vartheta]$ is the time of the control process beginning; hereinafter, for each $\tau \in [t_0, \vartheta]$, the symbol $x_\tau(\cdot)$ denotes the function on the interval $[-h, 0]$ defined by $x_\tau(\xi) = x(\tau + \xi)$, $\xi \in [-h, 0]$; $u(\tau)$ and $v(\tau)$ are control actions of the first and second players, respectively; $\mathbb{U}$ and $\mathbb{V}$ are known compacts of finite-dimensional spaces.

The first player aims to minimize the value γ of the quality index, while the second player aims to maximize it.

We assume that the following conditions hold:

(f_1) The functions $f(t, x, y, u, v) \in \mathbb{R}^n$, $f^0(t, x, y, u, v) \in \mathbb{R}$, $t \in [t_0, \vartheta]$, $x, y \in \mathbb{R}^n$, $u \in \mathbb{U}$, $v \in \mathbb{V}$, are continuous.

(f_2) For every $\alpha > 0$, there exists $\lambda_f > 0$ such that

$$\begin{aligned}&\|f(t,x,y,u,v) - f(t,x',y',u,v)\| \\ &\qquad + |f^0(t,x,y,u,v) - f^0(t,x',y',u,v)| \\ &\qquad\qquad \le \lambda_f \big(\|x - x'\| + \|y - y'\|\big)\end{aligned}$$

for any $t \in [t_0, \vartheta]$, $x, y, x', y' \in O(\alpha) = \{x \in \mathbb{R}^n \colon \|x\| \le \alpha\}$, $u \in \mathbb{U}$ and $v \in \mathbb{V}$.

(f_3) There exists a constant $c_f > 0$ such that

$$\|f(t,x,y,u,v)\| + |f^0(t,x,y,u,v)| \le c_f\big(1 + \|x\| + \|y\|\big)$$

for any $t \in [t_0, \vartheta]$, $x, y \in \mathbb{R}^n$, $u \in \mathbb{U}$ and $v \in \mathbb{V}$.

(f_4) For every $t \in [t_0, \vartheta]$ and $x, y, s \in \mathbb{R}^n$, we have

$$\begin{aligned}&\min_{u\in\mathbb{U}} \max_{v\in\mathbb{V}} \big(\langle f(t,x,y,u,v), s\rangle + f^0(t,x,y,u,v)\big) \\ &= \max_{v\in\mathbb{V}} \min_{u\in\mathbb{U}} \big(\langle f(t,x,y,u,v), s\rangle + f^0(t,x,y,u,v)\big)\end{aligned}$$

(σ) For every $\alpha > 0$, there exists $\lambda_\sigma > 0$ such that

$$|\sigma(z, w(\cdot)) - \sigma(z', w'(\cdot))| \le \lambda_\sigma\big(\|z - z'\| + \|w(\cdot) - w'(\cdot)\|_1\big)$$

for any $(z, w(\cdot)), (z', w'(\cdot)) \in P(\alpha)$, where

$$P(\alpha) = \{(z, w(\cdot)) \in \mathbb{R}^n \times \mathrm{PC} \colon \|z\| \le \alpha,\ \|w(\cdot)\|_\infty \le \alpha\}.$$

Let $(t, z, w(\cdot)) \in \mathbb{G}$. Denote by $\Lambda(t, z, w(\cdot))$ the set of functions $x(\cdot) \in \mathrm{PC}([t - h, \vartheta], \mathbb{R}^n)$ such that

$$x(\tau) = w(\tau - t),\ \tau \in [t - h, t), \quad x(\tau) = y(\tau),\ \tau \in [t, \vartheta],$$

where $y(\cdot) \in \mathrm{Lip}([t, \vartheta], \mathbb{R}^n)$ and $y(t) = z$.

Denote by $\mathcal{U}(t)$ and $\mathcal{V}(t)$ sets of measurable functions $u(\cdot) \colon [t, \vartheta] \mapsto \mathbb{U}$ and $v(\cdot) \colon [t, \vartheta] \mapsto \mathbb{V}$, respectively. It is known that, under the conditions above, for each $u(\cdot) \in \mathcal{U}(t)$ and $v(\cdot) \in \mathcal{V}(t)$, there exists a unique motion $x(\cdot)$ of system (1) that is a function $x(\cdot) \in \Lambda(t, z, w(\cdot))$ that satisfies equation (1) for almost every $\tau \in [t, \vartheta]$. The triple $\{x(\cdot), u(\cdot), v(\cdot)\}$ is called a control process realization. Note that this control process realization uniquely determines the value of quality index (2).

According to Krasovskii and Subbotin (1988) (see also Lukoyanov (2003) for time-delay systems), differential game (1), (2) is posed as follows.

By a control strategy of the first player, we mean an arbitrary function $U \colon \mathbb{G} \mapsto \mathbb{U}$. Let us fix $(t, z, w(\cdot)) \in \mathbb{G}$ and a partition of the interval $[t, \vartheta]$:

$$\Delta_\delta = \{\tau_j : \tau_1 = t, 0 < \tau_{j+1} - \tau_j \le \delta, j = \overline{1, l}, \tau_{l+1} = \vartheta\}. \quad (3)$$

The pair $\{U, \Delta_\delta\}$ defines a control law that forms a piecewise constant function $u(\cdot) \in \mathcal{U}(t)$ according to the following step-by-step rule:

$$u(\tau) = U(\tau_j, x(\tau_j), x_{\tau_j}(\cdot)), \quad \tau \in [\tau_j, \tau_{j+1}), \quad j = \overline{1, l}. \quad (4)$$

This control law together with $v(\cdot) \in \mathcal{V}(t)$ uniquely determine the control process realization $\{x(\cdot), u(\cdot), v(\cdot)\}$ and the value $\gamma = \gamma(t, z, w(\cdot); U, \Delta_\delta; v(\cdot))$ of quality index (2).

The guaranteed result of the strategy U is defined by

$$\rho_u(t, z, w(\cdot); U) = \lim_{\delta\downarrow 0} \sup_{\Delta_\delta} \sup_{v(\cdot)} \gamma(t, z, w(\cdot); U, \Delta_\delta; v(\cdot)). \quad (5)$$

The optimal guaranteed result of the first player is

$$\rho_u^\circ(t, z, w(\cdot)) = \inf_U \rho_u(t, z, w(\cdot); U). \quad (6)$$

A strategy U° of the first player is called optimal if

$$\rho_u(t, z, w(\cdot); U^\circ) = \rho_u^\circ(t, z, w(\cdot)).$$

Similarly, with the corresponding changes, for the second player, we define a control strategy $V \colon \mathbb{G} \mapsto \mathbb{V}$, control law $\{V, \Delta_\delta\}$ that forms a function $v(\cdot) \in \mathcal{V}(t)$ by

$$v(\tau) = V(\tau_j, x(\tau_j), x_{\tau_j}(\cdot)), \quad t \in [\tau_j, \tau_{j+1}), \quad j = \overline{1, l},$$

the guaranteed result of the strategy V

$$\rho_v(t, z, w(\cdot); V) = \lim_{\delta\downarrow 0} \inf_{\Delta_\delta} \inf_{u(\cdot)} \gamma(t, z, w(\cdot); u(\cdot); V, \Delta_\delta), \quad (7)$$

and the optimal guaranteed result

$$\rho_v^\circ(t, z, w(\cdot)) = \sup_V \rho_v(t, z, w(\cdot); V). \quad (8)$$

A strategy V° of the second player is called optimal if

$$\rho_v(t, z, w(\cdot); V^\circ) = \rho_v^\circ(t, z, w(\cdot)).$$

Due to definitions (5)–(8), we have

$$\rho_v^\circ(t, z, w(\cdot)) \le \rho_u^\circ(t, z, w(\cdot)), \quad (t, z, w(\cdot)) \in \mathbb{G}. \quad (9)$$

If the equality $\rho^\circ(t, z, w(\cdot)) := \rho_u^\circ(t, z, w(\cdot)) = \rho_v^\circ(t, z, w(\cdot))$ holds for any $(t, z, w(\cdot)) \in \mathbb{G}$, then $\rho^\circ \colon \mathbb{G} \mapsto \mathbb{R}$ is called the value functional of differential game (1), (2).

One can show (see, e.g., Krasovskii and Subbotin (1988)) that the value functional ρ° has the following properties:

(ρ_u) For every $(t, z, w(\cdot)) \in \mathbb{G}$, $\tau \in [t, \vartheta]$, $\varepsilon > 0$ and $v(\cdot) \in \mathcal{V}(t)$, there exists $u(\cdot) \in \mathcal{U}$ such that, for the control process realization $\{x(\cdot), u(\cdot), v(\cdot)\}$, we have

$$\begin{aligned}\rho^\circ(\tau, x(\tau), x_\tau(\cdot)) + \int_t^\tau f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi))\, \mathrm{d}\xi \\ \le \rho^\circ(t, z, w(\cdot)) + \varepsilon.\end{aligned}$$

(ρ_v) For every $(t, z, w(\cdot)) \in \mathbb{G}$, $\tau \in [t, \vartheta]$, $\varepsilon > 0$ and $u(\cdot) \in \mathcal{U}(t)$, there exists $v(\cdot) \in \mathcal{V}$ such that, for the control process realization $\{x(\cdot), u(\cdot), v(\cdot)\}$, we have

$$\begin{aligned}\rho^\circ(\tau, x(\tau), x_\tau(\cdot)) + \int_t^\tau f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi))\, \mathrm{d}\xi \\ \ge \rho^\circ(t, z, w(\cdot)) - \varepsilon.\end{aligned}$$

3. PROPERTIES OF THE TIME-DELAY SYSTEM

Taking the constant $c_f > 0$ form (f_3), we denote

$$\begin{aligned}F^\eta(x, y) = \{f \in \mathbb{R}^n \colon \|f\| \le c_f(1 + \|x\| + \|y\|) + \eta\}, \\ x, y \in \mathbb{R}^n, \quad \eta \ge 0. \quad (10)\end{aligned}$$

Let $(t, z, w(\cdot)) \in \mathbb{G}$ and $\eta \ge 0$. Denote by $X^\eta(t, z, w(\cdot))$ the set of functions $x(\cdot) \in \Lambda(t, z, w(\cdot))$ that satisfy the following time-delay differential inclusion:

$$\dot{x}(\tau) \in F^\eta(x(\tau), x(\tau - h)) \text{ for a.e. } \tau \in [t, \vartheta]. \quad (11)$$

Note that the set $X^\eta(t,z,w(\cdot))$ is not empty. In particular, for each $u(\cdot) \in \mathcal{U}(t)$ and $v(\cdot) \in \mathcal{V}(t)$, the motion $x(\cdot)$ of system (1) satisfies the inclusion

$$x(\cdot) \in X^0(t,z,w(\cdot)) \subset X^\eta(t,z,w(\cdot)), \quad \eta \geq 0. \tag{12}$$

Proposition 1. Let $(t,z,w(\cdot)) \in \mathbb{G}$ and $\eta \geq 0$. Then there exist $\alpha_X, \lambda_X > 0$ such that

$$(x(\tau), x_\tau(\cdot)) \in P(\alpha_X), \quad \|x(\tau) - x(\tau')\| \leq \lambda_X |\tau - \tau'|,$$

for any $\tau, \tau' \in [t,\vartheta]$ and $x(\cdot) \in X^\eta(t,z,w(\cdot))$.

Proposition 2. Let $(t,z,w(\cdot)) \in \mathbb{G}$, $t < \vartheta$ and $\eta \geq 0$. Let a sequence $x^k(\cdot) \in X^\eta(t,z,w(\cdot))$, $k \in \mathbb{N}$ be chosen. Then there exist a subsequence $x^{k_i}(\cdot)$ and a function $x^*(\cdot) \in X^\eta(t,z,w(\cdot))$ such that

$$\max_{\tau \in [t-h,\vartheta]} \|x^{k_i}(\tau) - x^*(\tau)\| \to 0 \text{ as } i \to \infty.$$

Proposition 3. Let $(t,z,w(\cdot)) \in \mathbb{G}$ and $\eta \geq 0$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x(\cdot) \in X^\eta(t,z,w(\cdot))$ and $\tau, \tau' \in [t,\vartheta]$: $|\tau - \tau'| \leq \delta$, we have:

$$\|x(\tau) - x(\tau')\| + \|x_\tau(\cdot) - x_{\tau'}(\cdot)\|_1 \leq \varepsilon.$$

Proposition 1 and 2 can be proved similar to Proposition 3.1 and Lemma 4.8 in Plaksin (2019), respectively. Proposition 3 one can proved, using approximation of $w(\cdot)$ by a Lipschitz function (see, e.g., (Natanson, 1960, p. 214)) and Proposition 1.

4. HJBI EQUATION

Following Kim (1999); Lukoyanov (2000), a functional $\varphi\colon \mathbb{G} \mapsto \mathbb{R}$ is called coinvariantly differentiable (ci-differentiable) at a point $(t,z,w(\cdot)) \in \mathbb{G}$, $t < \vartheta$ if there exist $\partial^{ci}_{t,w}\varphi(t,z,w(\cdot)) \in \mathbb{R}$ and $\nabla_z\varphi(t,z,w(\cdot)) \in \mathbb{R}^n$ such that, for every $v \in \mathbb{R}^n$, $x(\cdot) \in \Lambda(t,z,w(\cdot))$ and $\tau \in [t,\vartheta]$, the following relation holds:

$$\begin{aligned}\varphi(\tau,v,x_\tau(\cdot)) - \varphi(t,z,w(\cdot)) &= \partial^{ci}_{t,w}\varphi(t,z,w(\cdot))(\tau - t) \\ &+ \langle v - z, \nabla_z\varphi(t,z,w(\cdot))\rangle + o(|\tau - t| + \|v - z\|),\end{aligned} \tag{13}$$

where the function $x_\tau(\cdot) \in \mathrm{PC}$ is defined by $x_\tau(\xi) = x(\tau + \xi)$, $\xi \in [-h, 0]$, the value $o(\cdot)$ depends on the triplet $\{t, z, x(\cdot)\}$, and $o(\delta)/\delta \to 0$ as $\delta \to +0$. Then $\partial^{ci}_{t,w}\varphi(t,z,w(\cdot))$ is called the ci-derivative of φ with respect to $\{t, w(\cdot)\}$ and $\nabla_z\varphi(t,z,w(\cdot))$ is the gradient of φ with respect to z. Let us note that if φ does not depend on the functional variable $w(\cdot)$, then the definition of ci-differentiability coincides with the definition of differentiability of functions.

For differential game (1), (2), we define the Hamiltonian

$$\begin{aligned}H(t,x,y,s) = \min_{u \in \mathbb{U}} \max_{v \in \mathbb{V}} \big(&\langle f(t,x,y,u,v), s\rangle \\ &+ f^0(t,x,y,u,v)\big), \quad t \in [t_0,\vartheta], \quad x,y,s \in \mathbb{R}^n.\end{aligned} \tag{14}$$

and consider the following HJ equation

$$\begin{aligned}\partial^{ci}_{t,w}\varphi(t,z,w(\cdot)) + H(t,z,w(-h),\nabla_z\varphi(t,z,w(\cdot))) = 0,& \\ (t,z,w(\cdot)) \in \mathbb{G}, \quad t < \vartheta,&\end{aligned} \tag{15}$$

with the terminal condition

$$\varphi(\vartheta,z,w(\cdot)) = \sigma(z,w(\cdot)), \quad (\vartheta,z,w(\cdot)) \in \mathbb{G}. \tag{16}$$

Following Plaksin (2019), define the class of functionals in which we will search a solution of the problem (15), (16). Denote by Φ the set of functionals $\varphi = \varphi(t,z,w(\cdot)) \in \mathbb{R}$, $(t,z,w(\cdot)) \in \mathbb{G}$ which are continuous with respect to t and satisfy the following Lipschitz condition: for every $\alpha > 0$, there exists $\lambda_\varphi > 0$ such that

$$|\varphi(t,z,w(\cdot)) - \varphi(t,z',w'(\cdot))| \leq \lambda_\varphi\big(\|z - z'\| + \|w(\cdot) - w'(\cdot)\|_1\big).$$

for any $t \in [t_0,\vartheta]$, $(z,w(\cdot)),(z',w'(\cdot)) \in P(\alpha)$. The choice of this class is motivated, in particular, by the inclusion

$$\rho^\circ \in \Phi, \tag{17}$$

which can be shown by scheme of Lemma 4.1 in Plaksin (2019).

5. OPTIMAL STRATEGIES

Lemma 1. Let $\varphi \in \Phi$, $(t,z,w(\cdot)) \in \mathbb{G}$ and $\eta \geq 0$. For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $x(\cdot) \in X^\eta(t,z,w(\cdot))$ and $\tau, \tau' \in [t,\vartheta]$: $|\tau - \tau'| \leq \delta$, the following inequality holds:

$$|\varphi(\tau,x(\tau),x_\tau(\cdot)) - \varphi(\tau',x(\tau'),x_{\tau'}(\cdot))| \leq \varepsilon.$$

Proof. For the sake of a contradiction, suppose that there exist $\varepsilon > 0$ and $x^k(\cdot) \in X^\eta(t,z,w(\cdot))$, $\tau_k, \tau'_k \in [t,\vartheta]$: $|\tau_k - \tau'_k| \leq 1/k$, $k \in \mathbb{N}$ such that

$$|\varphi(\tau_k,x^k(\tau_k),x^k_{\tau_k}(\cdot)) - \varphi(\tau'_k,x^k(\tau'_k),x^k_{\tau'_k}(\cdot))| > \varepsilon, \quad k \in \mathbb{N}.$$

Without loss of generality, taking into account Proposition 2, we can suppose that there exist $\tau_* \in [t,\vartheta]$ and $x^*(\cdot) \in X^\eta(t,z,w(\cdot))$ such that

$$\begin{aligned}&\tau_k \to \tau_*, \quad \tau'_k \to \tau_*, \\ &\max_{\tau \in [t-h,\vartheta]} \|x^k(\tau) - x^*(\tau)\| \to 0,\end{aligned} \quad \text{as} \quad k \to \infty. \tag{18}$$

Due to Proposition 1 and the inclusion $\varphi \in \Phi$, there exist $k_1 > 0$ and $\lambda_\varphi > 0$ such that, for every $k > k_1$, we have

$$\begin{aligned}&|\varphi(\tau_k,x^k(\tau_k),x^k_{\tau_k}(\cdot)) - \varphi(\tau_*,x^*(\tau_*),x^*_{\tau_*}(\cdot))| \\ &\leq |\varphi(\tau_k,x^k(\tau_k),x^k_{\tau_k}(\cdot)) - \varphi(\tau_k,x^*(\tau_*),x^*_{\tau_*}(\cdot))| \\ &+ |\varphi(\tau_k,x^*(\tau_*),x^*_{\tau_*}(\cdot)) - \varphi(\tau_*,x^*(\tau_*),x^*_{\tau_*}(\cdot))| \\ &\leq \lambda_\varphi\big(\|x^k(\tau_k) - x^*(\tau_*)\| + \|x^k_{\tau_k}(\cdot) - x^*_{\tau_*}(\cdot)\|_1\big) + \varepsilon/4.\end{aligned}$$

According to Proposition 3 and (18), there exists $k_2 > 0$ such that, for every $k > k_2$, we derive

$$\begin{aligned}\|x^k(\tau_k) - x^k(\tau_*)\| + \|x^k_{\tau_k}(\cdot) - x^k_{\tau_*}(\cdot)\|_1 \leq \varepsilon/(8\lambda_\varphi), \\ \|x^k(\tau_*) - x^*(\tau_*)\| + \|x^k_{\tau_*}(\cdot) - x^*_{\tau_*}(\cdot)\|_1 \leq \varepsilon/(8\lambda_\varphi).\end{aligned}$$

Thus, for $k > \max\{k_1,k_2\}$, we obtain

$$|\varphi(\tau_k,x^k(\tau_k),x^k_{\tau_k}(\cdot)) - \varphi(\tau_*,x^*(\tau_*),x^*_{\tau_*}(\cdot))| \leq \varepsilon/2.$$

In a similar way, we can show that there exists $k_3 > 0$ such that, for every $k > k_3$, we have

$$|\varphi(\tau'_k,x^k(\tau'_k),x^k_{\tau'_k}(\cdot)) - \varphi(\tau_*,x^*(\tau_*),x^*_{\tau_*}(\cdot))| \leq \varepsilon/2.$$

Thus, for $k > \max\{k_1,k_2,k_3\}$, we conclude $\varepsilon < \varepsilon$. ■

Lemma 2. Let $\varphi \in \Phi$, $(t,z,w(\cdot)) \in \mathbb{G}$ and $\eta \geq 0$. There exists $\beta > 0$ such that, for every $x(\cdot) \in X^\eta(t,z,w(\cdot))$, the following estimate holds:

$$|\varphi(\tau,x(\tau),x_\tau(\cdot))| \leq \beta, \quad \tau \in [t,\vartheta]. \tag{19}$$

Proof. Since $\varphi \in \Phi$, there exists $\beta_0 > 0$ satisfying $|\varphi(\tau,0,\theta(\cdot) \equiv 0)| \leq \beta_0$, $\tau \in [t,\vartheta]$. Moreover, taking $\alpha_X > 0$ from Proposition 1, one can choose λ_φ such that

$$|\varphi(\tau, x(\tau), x_\tau(\cdot)) - \varphi(\tau, 0, \theta(\cdot))| \le \lambda_\varphi(\|x(\tau)\| + \|x_\tau(\cdot)\|_1),$$
$$\tau \in [t, \vartheta], \quad x(\cdot) \in X^\eta(t, z, w(\cdot)).$$

Thus, defining $\beta = \beta_0 + \lambda_\varphi(1+h)\alpha_X$, we obtain (19). ■

Lemma 3. Let $\varphi \in \Phi$ be ci-differentiable at every point $(t, z, w(\cdot)) \in \mathbb{G}$, $t < \vartheta$ and the inclusions $\partial^{ci}_{t,w}\varphi, \nabla_z\varphi \in \Phi$ hold. Let $(t, z, w(\cdot)) \in \mathbb{G}$ and $x^*(\cdot) \in X^0(t, z, w(\cdot))$. Then the function $\omega^*(\tau) = \varphi(\tau, x^*(\tau), x^*_\tau(\cdot))$, $\tau \in [t, \vartheta]$ is Lipschitz continuous and

$$\begin{aligned} \dot\omega^*(\tau) &= \partial^{ci}_{t,w}\varphi(\tau, x^*(\tau), x^*_\tau(\cdot)) \\ &+ \langle \dot x^*(\tau), \nabla_z\varphi(\tau, x^*(\tau), x^*_\tau(\cdot))\rangle \text{ for a.e. } \tau \in [t, \vartheta]. \end{aligned} \tag{20}$$

Proof. Taking $c_f > 0$ from condition (f_3) and $\alpha_X, \lambda_X > 0$ from Proposition 1, let us define $\eta = c_f(1 + 2\alpha_X)$. Then, due to inclusions $\partial^{ci}_{t,w}\varphi, \nabla_z\varphi \in \Phi$ and Lemma 2, there exists $\beta_{\partial\varphi}, \beta_{\nabla\varphi} > 0$ such that

$$|\partial^{ci}_{t,w}\varphi(\tau, x(\tau), x_\tau(\cdot))| \le \beta_{\partial\varphi}, \ \ |\nabla_z\varphi(\tau, x(\tau), x_\tau(\cdot))| \le \beta_{\nabla\varphi},$$
$$\tau \in [t, \vartheta], \quad x(\cdot) \in X^\eta(t, z, w(\cdot)). \tag{21}$$

Let us define the functions $x^k(\cdot) \in \Lambda(t, z, w(\cdot))$:

$$x^k(\tau) = k \int\limits_{\tau - 1/k}^{\tau} x^*(\max\{\xi, t\})\,\mathrm{d}\xi, \quad \tau \in [t, \vartheta], \quad k \in \mathbb{N}.$$

Then these functions are continuously differentiable and, for every $\tau \in [t, \vartheta]$ and $k \in \mathbb{N}$, satisfy relations

$$(x^k(\tau), x^k_\tau(\cdot)) \in P(\alpha_X), \quad \|\dot x^k(\tau)\| \le \lambda_X. \tag{22}$$

Moreover, one can show that

$$\max_{\tau \in [t,\vartheta]} \|x^*(\tau) - x^k(\tau)\| \to 0 \text{ as } k \to \infty. \tag{23}$$

and, using definition (11) of the set $X^0(t, z, w(\cdot))$ and in accordance with the choice of the number η, for all $\tau \in [t, \vartheta]$, we derive

$$\|\dot x^k(\tau)\| \le k \int\limits_{\max\{\tau - 1/k, t\}}^{\tau} \|\dot x^*(\xi)\|\,\mathrm{d}\xi$$
$$\le k \int\limits_{\max\{\tau - 1/k, t\}}^{\tau} c_f\big(1 + \|x^*(\xi)\| + \|x^*(\xi - h)\|\big)\,\mathrm{d}\xi \le \eta.$$

It means that

$$x^k(\cdot) \in X^\eta(t, z, w(\cdot)), \quad k \in \mathbb{N}. \tag{24}$$

Define the functions $\omega^k(\tau) = \varphi(\tau, x^k(\tau), x^k_\tau(\cdot))$, $\tau \in [t, \vartheta]$, $k \in \mathbb{N}$. Then, according to the inclusion $\varphi \in \Phi$ and (22), there exists $\lambda_\varphi > 0$ such that, for every $\tau \in [t, \vartheta]$, we have

$$|\omega^*(\tau) - \omega^k(\tau)| \le \lambda_\varphi\big(\|x^*(\tau) - x^k(\tau)\| + \|x^*_\tau(\cdot) - x^k_\tau(\cdot)\|_1\big).$$

Then from (23) we obtain

$$\max_{\tau \in [t,\vartheta]} \|\omega^*(\tau) - \omega^k(\tau)\| \to 0 \quad \text{as} \quad k \to \infty. \tag{25}$$

Taking into account ci-differentiability of φ, let us calculate right derivatives of the functions $\omega^k(\cdot)$ as follows

$$\begin{aligned} \mathrm{d}^+\omega^k(\tau)/\,\mathrm{d}\tau &= \lim_{\xi \to \tau + 0} \frac{\omega^k(\xi) - \omega^k(\tau)}{\xi - \tau} \\ &= \partial^{ci}_{t,w}\varphi(\tau, x^k(\tau), x^k_\tau(\cdot)) \\ &+ \langle \dot x^k(\tau), \nabla_z\varphi(\tau, x^k(\tau), x^k_\tau(\cdot))\rangle, \quad \tau \in (t, \vartheta). \end{aligned} \tag{26}$$

Due to the inclusions $\partial^{ci}_{t,w}\varphi, \nabla_z\varphi \in \Phi$, Lemma 1 and continuous differentiability of $x^k(\cdot)$, we obtain that the right-hand side of this equation is continuous on (t, ϑ). Consequently, the function $\mathrm{d}^+\omega^k(\tau)/\,\mathrm{d}\tau$ is also continuous on (t, ϑ). Then one can show that the function $\omega(\cdot)$ is differentiable on (t, ϑ) and $\dot\omega^k(\tau) = \mathrm{d}^+\omega^k(\tau)/\,\mathrm{d}\tau$, $\tau \in (t, \vartheta)$. Thus, from (21), (22), (24), (26), we derive

$$|\dot\omega^k(\tau)| \le \beta_{\partial\varphi} + \lambda_X\beta_{\nabla\varphi} := \lambda_\omega, \quad \tau \in (t, \vartheta).$$

It means that

$$|\omega^k(\tau) - \omega^k(\tau')| \le \lambda_\omega|\tau - \tau'|, \quad \tau, \tau' \in [t, \vartheta].$$

Passing to the limit in this estimate as $k \to \infty$, considering (25), we obtain that the function $\omega^*(\cdot)$ is Lipschitz continuous. Equality (20) is obtained similar to (26) at the points of differentiability of functions $\omega^*(\cdot)$ and $x^*(\cdot)$. ■

Let us consider the following players control strategies:

$$\begin{aligned} U^\circ(t, z, w(\cdot)) &\in \underset{u \in \mathbb{U}}{\operatorname{argmin}} \max_{v \in \mathbb{V}} \chi(t, z, w(\cdot), u, v), \\ V^\circ(t, z, w(\cdot)) &\in \underset{v \in \mathbb{V}}{\operatorname{argmax}} \min_{u \in \mathbb{U}} \chi(t, z, w(\cdot), u, v), \end{aligned} \tag{27}$$

where $(t, z, w(\cdot)) \in \mathbb{G}$ and

$$\begin{aligned} \chi(t, z, w(\cdot), u, v) &= \langle f(t, z, w(-h), u, v), \nabla\varphi_z(t, z, w(\cdot))\rangle \\ &+ f^0(t, z, w(-h), u, v). \end{aligned} \tag{28}$$

Theorem 1. Let a functional $\varphi \in \Phi$ be ci-differentiable at every point $(t, z, w(\cdot)) \in \mathbb{G}$, $t < \vartheta$, satisfies HJ equation (15) with terminal condition (16) and the inclusions $\partial^{ci}_{t,w}\varphi, \nabla_z\varphi \in \Phi$ hold. Then the control strategies U° and V° defined by (27) are optimal, and φ is the value functional of differential game (1), (2).

Proof. The proof is carried out by the scheme from Lukoyanov (2003) (see also Gomoyunov and Plaksin (2018)). Let $(t, z, w(\cdot)) \in \mathbb{G}$. If $t = \vartheta$, then the validity of the theorem follows from the definition of ρ° and (16). Let $t < \vartheta$. In accordance with (6), (8) and (9), for proving the theorem, it is sufficient to show that

$$\rho_u(t, z, w(\cdot); U^\circ) \le \varphi(t, z, w(\cdot)) \le \rho_v(t, z, w(\cdot); V^\circ). \tag{29}$$

Let us prove the first inequality. Due to (5), we should show that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that the following statement is valid. Let Δ_δ be a partition (3). Then, for the control process realization $\{x(\cdot), u(\cdot), v(\cdot)\}$ generated by the control law of the first player $\{U^\circ, \Delta_\delta\}$ and $v(\cdot) \in \mathcal{V}(t)$, the value $\gamma = \gamma(t, z, w(\cdot); U^\circ, \Delta_\delta; v(\cdot))$ of quality index (2) satisfies the inequality

$$\begin{aligned} \gamma = \sigma(x(\vartheta), x_\vartheta(\cdot)) &+ \int_t^\vartheta f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi))\,\mathrm{d}\xi \\ &\le \varphi(t, z, w(\cdot)) + \varepsilon. \end{aligned} \tag{30}$$

Due to the inclusion $\nabla_z\varphi \in \Phi$ and Lemma 2, there exists $\beta_{\nabla\varphi} > 0$ such that, for every $x(\cdot) \in X^0(t, z, w(\cdot))$, we have

$$|\nabla_z\varphi(\tau, x(\tau), x_\tau(\cdot))| \le \beta_{\nabla\varphi}, \quad \tau \in [t, \vartheta]. \tag{31}$$

Denote

$$\varepsilon_* = \varepsilon/(2(\vartheta - t)). \tag{32}$$

Let $-h < \xi_1 < \xi_2 < \ldots < \xi_k < 0$ are discontinuity points of the function $w(\cdot)$. Define the sets

$$I_i = [\xi_i + t + h, \xi_{i+1} + t + h) \cap [t, \vartheta], \quad i \in \overline{1, k-1}$$
$$I_k = [\xi_l + t + h, t + h) \cap [t, \vartheta], \quad I_{k+1} = [\min\{t + h, \vartheta\}, \vartheta].$$

Then, according to piecewise continuity of $w(\cdot)$, Proposition 1 and condition (f_1), there exists $\delta_f > 0$ such that, for every $x(\cdot) \in X^0(t, z, w(\cdot))$, $i \in \overline{1, k+1}$, $\tau, \tau' \in I_i$, $u \in \mathbb{U}$ and $v \in \mathbb{V}$, the following estimates hold:

$$\begin{aligned} &\|f(\tau, x(\tau), x(\tau - h), u, v) \\ &\qquad - f(\tau', x(\tau'), x(\tau' - h), u, v)\| \leq \varepsilon_*/\beta_{\nabla\varphi}, \\ &|f^0(\tau, x(\tau), x(\tau - h), u, v) \\ &\qquad - f^0(\tau', x(\tau'), x(\tau' - h), u, v)| \leq \varepsilon_*. \end{aligned} \tag{33}$$

Moreover, there exists $\beta_f > 0$ such that

$$\begin{gathered} \|f(\tau, x(\tau), x(\tau - h), u, v)\| \leq \beta_f, \\ |f^0(\tau, x(\tau), x(\tau - h), u, v)| \leq \beta_f, \\ \tau \in [t, \vartheta], \quad x(\cdot) \in X^0(t, z, w(\cdot)), \quad u \in \mathbb{U}, \quad v \in \mathbb{V}. \end{gathered} \tag{34}$$

Due to Lemma 1 and the inclusions $\varphi, \partial^{ci}_{t,w}\varphi, \nabla_z\varphi \in \Phi$, there exists $\delta_\varphi > 0$ such that, for every $x(\cdot) \in X^0(t, z, w(\cdot))$ and $\tau, \tau' \in [t, \vartheta]$: $|\tau - \tau'| \leq \delta_\varphi$, we obtain

$$\begin{gathered} |\varphi(\tau, x(\tau), x_\tau(\cdot)) - \varphi(\tau', x(\tau'), x_{\tau'}(\cdot))| \leq \varepsilon/(4(k+1)), \\ |\partial^{ci}_{t,w}\varphi(\tau, x(\tau), x_\tau(\cdot)) - \partial^{ci}_{t,w}\varphi(\tau', x(\tau'), x_{\tau'}(\cdot))| \leq \varepsilon_*, \\ |\nabla_z\varphi(\tau, x(\tau), x_\tau(\cdot)) - \nabla_z\varphi(\tau', x(\tau'), x_{\tau'}(\cdot))| \leq \varepsilon_*/\beta_f. \end{gathered} \tag{35}$$

Define

$$\delta = \min\{\delta_f, \delta_\varphi, \varepsilon/(4\beta_f(k+1))\}. \tag{36}$$

Let us show that this δ satisfies the statement above. Let Δ_δ be a partition (3) and the control process realization $\{x(\cdot), u(\cdot), v(\cdot)\}$ be generated by the control law $\{U^\circ, \Delta_\delta\}$ and $v(\cdot) \in \mathcal{V}(t)$. Define the function

$$\begin{aligned} \omega(\tau) &= \varphi(\tau, x(\tau), x_\tau(\cdot)) \\ &+ \int_t^\tau f^0(\xi, x(\xi), x(\xi - h), u(\xi), v(\xi))\,\mathrm{d}\xi, \quad \tau \in [t, \vartheta]. \end{aligned} \tag{37}$$

Then, taking into account terminal condition (16), for proving (30), it is sufficient to show that

$$\omega(\vartheta) \leq \omega(t) + \varepsilon. \tag{38}$$

Denote by J the set of $j \in \overline{1, l}$ such that there exists $i \in \overline{1, k+1}$ such that $[\tau_j - h, \tau_{j+1} - h) \subset I_i$. Note that, $|J| \geq l - k - 1$. Then, from (34)–(37), we derive

$$\omega(\vartheta) - \omega(t) \leq \sum_{j \in J} (\omega(\tau_{j+1}) - \omega(\tau_j)) + \varepsilon/2. \tag{39}$$

According to Lemma 3, the function $\omega(\cdot)$ is Lipschitz continuous and, taking into account system (1) and (28), for almost every $\tau \in [t, \vartheta]$, satisfies the equation

$$\dot\omega(\tau) = \partial^{ci}_{t,w}\varphi(\tau, x(\tau), x_\tau(\cdot)) + \chi(\tau, x(\tau), x_\tau(\cdot), u(\tau), v(\tau)).$$

For $j \in J$ and $\tau \in [\tau_j, \tau_{j+1})$, in accordance with (31), (33)–(36), we derive

$$\begin{aligned} &\partial^{ci}_{t,w}\varphi(\tau, x(\tau), x_\tau(\cdot)) + \chi(\tau, x(\tau), x_\tau(\cdot), u(\tau), v(\tau)) \\ &\quad\leq \partial^{ci}_{t,w}\varphi(\tau_j, x(\tau_j), x_{\tau_j}(\cdot)) \\ &\quad + \chi(\tau_j, x(\tau_j), x_{\tau_j}(\cdot), u(\tau), v(\tau)) + 4\varepsilon_*. \end{aligned}$$

Due to (4), (14), (27), we have

$$\begin{aligned} &\chi(\tau_j, x(\tau_j), x_{\tau_j}(\cdot), u(\tau), v(\tau)) \\ &\leq \max_{v \in \mathbb{V}} \chi(\tau_j, x(\tau_j), x_{\tau_j}(\cdot), U^\circ(\tau_j, x(\tau_j), x_{\tau_j}(\cdot)), v) \\ &= H(\tau_j, x(\tau_j), x(\tau_j - h), \nabla\varphi(\tau_j, x(\tau_j), x_{\tau_j}(\cdot))). \end{aligned}$$

Thus, taking into account (15), we obtain

$$\dot\omega(\tau) \leq \varepsilon_* \text{ for a.e. } \tau \in [\tau_j, \tau_{j+1}], \quad j \in J.$$

Then, from (32), (39), we conclude (38). The first inequality in (29) is proved. Due to (f_4), the second inequality in (29) can be proved in a similar way. ∎

6. CI-DIFFERENTIABILITY PROPERTIES OF THE VALUE FUNCTIONAL

Lemma 4. For every $(t, z, w(\cdot)) \in \mathbb{G}$, $t < \vartheta$ and $s \in \mathbb{R}^n$, there exist $\tau_* \in (t, \vartheta]$ and $x^*(\cdot) \in X^0(t, z, w(\cdot))$ such that, for every $\tau \in [t, \tau_*]$, the following inequality holds:

$$\begin{aligned} &\rho^\circ(\tau, x^*(\tau), x^*_\tau(\cdot)) \\ &+ \int_t^\tau \big(H(\xi, x^*(\xi), x^*(\xi - h), s) - \langle \dot x^*(\xi), s\rangle\big)\,\mathrm{d}\xi \\ &\qquad\qquad \leq \rho^\circ(t, z, w(\cdot)). \end{aligned} \tag{40}$$

Proof. Since $w(\cdot) \in \mathrm{PC}$, then there exists $\tau_* \in (t, \vartheta]$ such that $w(\cdot)$ is continuous on $[-h, \tau_* - t - h]$. Let $k \in \mathbb{N}$. Denote $\tau_j = t + (\tau_* - t)j/k$, $j \in \overline{0, k}$. According to (14) and (ρ_u), let us define control process realizations $\{x^k(\cdot), u^k(\cdot), v^k(\cdot)\}$ such that, for every $j \in \overline{0, k-1}$, we have

$$v^k(\tau) = v^k_j, \quad \tau \in [\tau_j, \tau_{j+1}), \tag{41}$$

where $v^k_j \in \mathbb{V}$ is defined by

$$\begin{aligned} &H(\tau_j, x^k(\tau_j), x^k(\tau_j - h), s) \\ &= \min_{u \in \mathbb{U}} \big(\langle f(\tau_j, x^k(\tau_j), x^k(\tau_j - h), u, v^k_j), s\rangle \\ &\qquad + f^0(\tau_j, x^k(\tau_j), x^k(\tau_j - h), u, v^k_j)\big), \end{aligned} \tag{42}$$

and

$$\begin{aligned} &\rho^\circ(\tau_{j+1}, x^k(\tau_{j+1}), x^k_{\tau_{j+1}}(\cdot)) \\ &+ \int_{\tau_j}^{\tau_{j+1}} f^0(\xi, x^k(\xi), x^k(\xi - h), u^k(\xi), v^k(\xi))\,\mathrm{d}\xi \\ &\qquad \leq \rho^\circ(\tau_j, x^k(\tau_j), x^k_{\tau_j}(\cdot)) + (\tau_{j+1} - \tau_j)/k. \end{aligned} \tag{43}$$

According to (12) and Proposition 2, without loss of generality, we can suppose that there exists $x^*(\cdot) \in X^0(t, z, w(\cdot))$ such that

$$\max_{\tau \in [t, \vartheta]} \|x^*(\tau) - x^k(\tau)\| \to 0 \quad \text{as} \quad k \to \infty. \tag{44}$$

Let us fix $\varepsilon > 0$. Due to (f_1) and Lemma 1, taking into account the continuity of $w(\cdot)$ on $[-h, \tau_* - t - h]$, Proposition 1 and (17), there exists $k_1 > 0$ such that, for every $k \geq k_1$, $x(\cdot) \in X^0(t, z, w(\cdot))$, $u \in \mathbb{U}$, $v \in \mathbb{V}$, $j \in \overline{0, k-1}$ and $\tau \in [\tau_j, \tau_{j+1}]$, we have

$$\begin{aligned} &|\langle f(\tau_j, x(\tau_j), x(\tau_j - h), u, v), s\rangle \\ &\quad - \langle f(\tau, x(\tau), x(\tau - h), u, v), s\rangle| \leq \varepsilon/(16(\tau_* - t)), \\ &|f^0(\tau_j, x(\tau_j), x(\tau_j - h), u, v) \\ &\quad - f^0(\tau, x(\tau), x(\tau - h), u, v)| \leq \varepsilon/(16(\tau_* - t)), \\ &|\rho^\circ(\tau_j, x(\tau_j), x_{\tau_j}(\cdot)) - \rho^\circ(\tau, x(\tau), x_\tau(\cdot))| \leq \varepsilon/6. \end{aligned} \tag{45}$$

Then, in accordance with (14), we derive

$$|H(\tau_j, x(\tau_j), x(\tau_j - h), s) \\ -H(\tau, x(\tau), x(\tau - h), s)| \leq \varepsilon/(8(\tau_* - t)). \tag{46}$$

Moreover, according to (f_1) and Proposition 1, there exists $\beta_f > 0$ such that

$$|f^0(\tau, x(\tau), x(\tau - h), u, v)| \leq \beta_f, \\ \tau \in [t, \tau_*], \quad x(\cdot) \in X^0(t, z, w(\cdot)), \quad u \in \mathbb{U}, \quad v \in \mathbb{V}. \tag{47}$$

Due to (17), (f_1), (14) and (44), there exists $k_2 > 0$ such that, for every $k \geq k_2$, we have

$$\rho^\circ(\tau, x^*(\tau), x^*_\tau(\cdot)) \\ + \int_t^\tau \big(H(\xi, x^*(\xi), x^*(\xi - h), s) - \langle \dot{x}^*(\xi), s\rangle\big)\, \mathrm{d}\xi \\ \leq \rho^\circ(\tau, x^k(\tau), x^k_\tau(\cdot)) \\ \int_t^\tau \big(H(\xi, x^k(\xi), x^k(\xi - h), s) - \langle \dot{x}^k(\xi), s\rangle\big)\, \mathrm{d}\xi + \varepsilon/6. \tag{48}$$

From (1), (42), (45), for every $j \in \overline{0, k-1}$ and almost every $\xi \in [\tau_j, \tau_{j+1})$, we derive

$$H(\xi, x^k(\xi), x^k(\xi - h), s) - \langle \dot{x}^k(\xi), s\rangle \\ \leq H(\tau_j, x^k(\tau_j), x^k(\tau_j - h), s) \\ -\langle f(\tau_j, x^k(\tau_j), x^k(\tau_j - h), u^k(\xi), v^k_j), s\rangle + 3\varepsilon/(16(\tau_* - t)) \\ \leq f^0(\tau_j, x^k(\tau_j), x^k(\tau_j - h), u^k(\xi), v^k_j) + 3\varepsilon/(16(\tau_* - t)) \\ \leq f^0(\xi, x^k(\xi), x^k(\xi - h), u^k(\xi), v^k_j) + \varepsilon/(4(\tau_* - t)).$$

Then, taking into account (47) and defining $k_3 = 4\beta_f(\tau_* - t)/\varepsilon$, for $k \geq k_3$, $j \in \overline{0, k-1}$ and $\tau \in [\tau_j, \tau_{j+1})$, we have

$$\int_t^\tau \big(H(\xi, x^k(\xi), x^k(\xi - h), s) - \langle \dot{x}^k(\xi), s\rangle\big)\, \mathrm{d}\xi \\ \leq \int_t^\tau f^0(\xi, x^k(\xi), x^k(\xi - h), u^k(\xi), v^k_j)\, \mathrm{d}\xi + \varepsilon/4 \\ \leq \int_t^{\tau_j} f^0(\xi, x^k(\xi), x^k(\xi - h), u^k(\xi), v^k_j)\, \mathrm{d}\xi + \varepsilon/2. \tag{49}$$

Thus, from (41), (43), (45), (48), (49), for

$$k \geq \max\{6(\tau_* - t)/\varepsilon, k_1, k_2, k_3\},$$

we obtain

$$\rho^\circ(\tau, x^*(\tau), x^*_\tau(\cdot)) \\ + \int_t^\tau \big(H(\xi, x^*(\xi), x^*(\xi - h), s) - \langle \dot{x}^*(\xi), s\rangle\big)\, \mathrm{d}\xi \\ \leq \rho^\circ(t, z, w(\cdot)) + \varepsilon.$$

It holds for any $\varepsilon > 0$, therefore we can put $\varepsilon = 0$. ■

Lemma 5. For every $(t, z, w(\cdot)) \in \mathbb{G}$, $t < \vartheta$ and $s \in \mathbb{R}^n$, there exist $\tau_* \in (t, \vartheta]$ and $x^*(\cdot) \in X^0(t, z, w(\cdot))$ such that, for every $\tau \in [t, \tau_*]$, the following inequality holds:

$$\rho^\circ(\tau, x^*(\tau), x^*_\tau(\cdot)) \\ + \int_t^\tau \big(H(\xi, x^*(\xi), x^*(\xi - h), s) - \langle \dot{x}^*(\xi), s\rangle\big)\, \mathrm{d}\xi \\ \geq \rho^\circ(t, z, w(\cdot)).$$

Proof. The Lemma can be proved similar to Lemma 4, using (ρ_v) and (f_4). ■

Theorem 2. Let the value functional ρ° of differential game (1), (2) be ci-differentiable at a point $(t, z, w(\cdot)) \in \mathbb{G}$, $t < \vartheta$. Then it satisfies HJ equation (15) at this point.

Proof. Due to Lemma 4, defining $s = \nabla_z \rho^\circ(t, z, w(\cdot))$, let us take $\tau_* \in (t, \vartheta]$ and $x^*(\cdot) \in X^0(t, z, w(\cdot))$. By the definition of ci-differentiability of ρ°, we have

$$\rho^\circ(\tau, x^*(\tau), x^*_\tau(\cdot)) - \rho^\circ(t, z, w(\cdot)) = \partial^{ci}_{t,w}\rho^\circ(t, z, w(\cdot))(\tau - t) \\ + \langle x^*(\tau) - z, \nabla_z \rho^\circ(t, z, w(\cdot))\rangle + o(\tau - t), \quad \tau \in [t, \vartheta].$$

Then, using (40), for $\tau \in [t, \tau_*]$, we derive

$$\partial^{ci}_{t,w}\rho^\circ(t, z, w(\cdot))(\tau - t) + o(\tau - t) \\ + \int_t^\tau H(\xi, x^*(\xi), x^*(\xi - h), \nabla_z \rho^\circ(t, z, w(\cdot)))\, \mathrm{d}\xi \leq 0.$$

Dividing this inequality by $\tau - t$, passing to the limit as $\tau \to t + 0$, taking into account (f_1) and (14), we get

$$\partial^{ci}_{t,w}\rho^\circ(t, z, w(\cdot)) + H(t, z, w(-h), \nabla_z \rho^\circ(t, z, w(\cdot))) \leq 0.$$

In a similar way, using Lemma 5, we can obtain the opposite inequality. ■

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